Synchronization of finite-state pulse-coupled oscillators

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Abstract

We propose a novel cellular automaton (CA) model for n-state pulse-coupled inhibitory oscillators, which we call the firefly cellular automaton (FCA), and study the emergence of synchrony on various underlying graphs. FCA is strictly related to other two CA models for excitable media, namely, the cyclic cellular automaton (CCA) and the Greenberg-Hastings model (GHM). On Z, all three models belong to the same universality class due to a common n-coloring on Z that exhibits wide range of dynamical behavior, but the FCA shows a critical behavior on all other n-dimensional lattice. We also give a 4-color FCA into a continuous setting and obtain a clock synchronization algorithm for distributed networks on an arbitrary connected graph $G = (V, E)$, which runs in $O(|V|)$ times.

1 Introduction

Complex systems are dynamical systems of large number of locally interacting agents, where various large-scale behavior often emerges from non-linear local interactions. Generalized cellular automaton (GCA) is a discrete formulation for such systems, which was first invented by Von Neumann in 1950’s as an effort to emulate self-replicating organisms on a lattice universe. A GCA is defined by a triple $(G, X_v, \tau)$, where $G = (V, E)$ is a fixed simple graph, $X_v : V \to \mathbb{Z}^m$ is a $m$-coloring for some fixed integer $m \geq 2$, and $\tau$ is a locally defined deterministic transition map which maps a given $s$-coloring $X_v$ to the next $X_v$. The iteration of $\tau$ then generates a discrete-time deterministic dynamical system, and the limiting behavior of the orbit $(X_v^n)_{n=0}$ is of interest.

An extensively studied class of complex systems are excitable media, due to their ubiquitous appearance in many systems such as nerve cells, muscle cells, cardiac function, chemical reaction, and coupled oscillators. Characteristic of these systems is self-organization; they exhibit wide range of emergent behaviors such as traveling waves, spontaneous or spatial pattern formations, and may synchronize to equilibrium or fluctuate in non-equilibrium with complex dynamic patterns. While continuous reaction-diffusion systems are often used to describe such models, two rigorously studied GCA models for excitable media are the cyclic cellular automaton (CCA) and the Greenberg-Hastings model (GHM). We propose another class of GCA models which we call the firefly cellular automaton (FCA), which in particular models the coupled oscillators, and study the emergence of synchrony.

2 The firefly cellular automaton

Let $G = (V, E)$ be a graph and let $Z_n \subseteq \mathbb{Z}/(Z_n)$. A n-coloring on $G$ is a function $X : V \to Z_n$. Define the blanking color $h(x) = \lfloor \frac{x}{n} \rfloor$. Time evolution is given by the $X \to X^\tau$ defined by

$$X^\tau(v) = \begin{cases} X(v) & \text{if } X(v) > h(v) \text{ and } v \text{ is adj.} \\ X(v) + 1 & \text{otherwise} \end{cases}$$

Iteration of this map generates $s$-periodic firefly cellular automaton on $G$. We write $(G, X_0)$ for a $s$-periodic firefly network starting from a given initial configuration $X_0$. We say $X_0$ synchronizes if $X_0 = \tau(t mod s)$ converges to a constant function.

3 FCA on finite graphs

Theorem 1. For any $s \geq 3$, if $G = (V, E)$ is a path of s vertices and $X_0$ is any $s$-configuration, then $(X_0)_V$ synchronizes.

Theorem 2. Let $G = (V, E)$ be a finite tree. Then for $s \in \{3, 4, 5, 6\}$, arbitrary $s$-coloring $X_0 : V \to Z_s$ on $G$ synchronizes if and only if $T$ has max degree $< s$.

4 FCA on $\mathbb{Z}$

Theorem 4. Let $G = (\mathbb{Z}, E)$ be the integer lattice with nearest-neighbor edges, for $s \geq 3$, and let $X_0$ be drawn from the uniform product measure on $Z_s^n$. Then for any $s \in \mathbb{Z}$, we have

$$P(X_s = X_{s+1}) = O(e^{-1/2|s|}).$$

Proof. Uses a comparison with an embedded particle system and Spurr Anderson theorem for the survival probability of random walks with s i.i.d. increments.

References


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